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ASYMPTOTIC SOLUTION OF INITIAL BOUNDARY-VALUE PROBLEMS FOR HYPERBOLIC SYSTEMS†

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CONTENTS

A theory is developed for the derivation of formal asymptotic solutions for initial boundary-value problems for equations of the form

$$A^0\frac{\partial \mathbf{u}}{\partial t}+\sum_{\nu=1}^{n}A^\nu\frac{\partial \mathbf{u}}{\partial x_\nu}+\lambda B\mathbf{u}+C\mathbf{u}=\mathbf{f}(t,\mathbf{X};\lambda),$$

where A^0, A^ν, B, and C are $m\times m$ matrix functions of t and $\mathbf{X}=(x_1,\ldots,x_n)$, $\mathbf{u}(t,\mathbf{X};\lambda)$ is an m-component column vector, and λ is a large positive parameter. Our procedure is to consider a formal asymptotic solution of the form

$$\mathbf{u}(t,\mathbf{X};\lambda)\sim \mathrm{e}^{\mathrm{i}\lambda s(t,\mathbf{X})}\sum_{j=0}^{\infty}(\mathrm{i}\lambda)^{-j}\,\mathbf{z}_j(t,\mathbf{X}).$$

Substitution of this formal solution into the equation yields, for the function $s(t,\mathbf{X})$, a first order partial differential equation which can be solved by the method of characteristics. If the coefficient matrices satisfy certain conditions then we obtain, for the functions $\mathbf{z}_j(t,\mathbf{X})$, linear systems of ordinary differential equations called *transport equations* along space-time curves called *rays*. They may be solved explicitly under suitable conditions. A proof is presented of the asymptotic nature of the formal solution when the coefficient matrices and initial data for $\mathbf{u}$ are appropriately chosen. The problem of reflexion and refraction at an interface is considered.

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1. Introduction

The purpose of this paper is tho develop a theory for the derivation of formal asymptotic solutions for initial boundary-value problems for hyperbolic equations of the form

$$A^0(t,\mathbf{X})\frac{\partial \mathbf{u}}{\partial t}+\sum_{\nu=1}^{n}A^\nu(t,\mathbf{X})\frac{\partial \mathbf{u}}{\partial x_\nu}+\lambda B(t,\mathbf{X})\,\mathbf{u}+C(t,\mathbf{X})\,\mathbf{u}=\mathbf{f}(t,\mathbf{X};\lambda). \tag{1}$$

Here A^0, A^ν $(\nu=1,\ldots,n)$, B, and C are $m\times m$ smooth matrix functions of t and $\mathbf{X}=(x_1,\ldots,x_n)$, $\mathbf{u}(t,\mathbf{X})$ is an m-component vector, and λ is a large positive parameter. The condition of hyperbolicity plays no role in the formal theory. However, we shall see that it is important for the rigorous discussion of § 3·2. A distinguishing feature of the problem we shall treat is that the parameter λ appears in the equation as a multiplicative factor of an undifferentiated term as well as in the initial data $\mathbf{u}(0,\mathbf{X})=\mathbf{g}(\mathbf{X},\lambda)$. If $B=C=0$, then (1) is 'non-dispersive'. If $B=0$ and $C\neq 0$ then we say that (1) is *weakly dispersive*, i.e. only an undifferentiated term appears which is not multiplied by the large parameter λ. If $B\neq 0$ then we say that (1) is *strongly dispersive*, i.e. an undifferentiated term is multiplied by the large parameter.

An asymptotic method for treating the dispersive hyperbolic equation

$$u_{tt}-c^2(\mathbf{X})\,\Delta u+\lambda^2 b^2(\mathbf{X})\,u=0 \tag{2}$$

was developed by Lewis (1964). Further investigations were carried out by Bleistein (1965). Our procedure is a direct extension of this method. The vector nature of the equation that we consider is the primary difficulty that must be confronted. This difficulty is resolved by standard techniques of linear algebra.

We have previously (Granoff & Lewis 1966) obtained the leading term of the asymptotic expansion as $\lambda\to\infty$ of the solution of some problems of the form (1) with constant coefficients which can be solved exactly. From the form of those asymptotic expressions it is reasonable to seek, in those problems that cannot be solved exactly, a formal asymptotic solution of the form

$$\mathbf{u}(t,\mathbf{X};\lambda)\sim e^{i\lambda s(t,\mathbf{X})}\sum_{j=0}^{\infty}(i\lambda)^{-j}\,\mathbf{z}_j(t,\mathbf{X})\quad(\lambda\to\infty). \tag{3}$$

Substitution of this formal solution into (1) yields, for the *phase function* $s(t,\mathbf{X})$, a nonlinear first order partial differential equation, which may be solved by the method of characteristics (Courant & Hilbert 1962). The corresponding characteristic curves in space-time are called *rays*. In the nondispersive or weakly dispersive cases the rays lie on the characteristic hypersurface and are the 'bicharacteristics' of (1). In the strongly dispersive case they do not lie on the characteristic hypersurfaces and this leads to the more interesting features of the theory. If the coefficient matrices satisfy suitable conditions then we also obtain, for the *amplitude functions* $\mathbf{z}_j(t,\mathbf{X})$, recursive first order linear systems of ordinary differential equations along the rays which can be solved. These systems of equations are called *transport equations*. The *ray theory* outlined above is presented in § 2.

Lewis (1965) develops an asymptotic theory for an integro-differential equation of the form

$$\frac{\partial \mathbf{v}}{\partial t}+\sum_{\nu=1}^{n}A^\nu(t,\mathbf{X})\frac{\partial \mathbf{u}}{\partial x_\nu}=0,\quad \mathbf{v}(t,\mathbf{X})=\int_{-\infty}^{\infty}F(\tau)\,\mathbf{u}(t-\tau,\mathbf{X})\,d\tau. \tag{4}$$

Equations of this type occur in electromagnetic theory. He shows that, for a special choice of $F(\tau)$, (4) reduces to a partial differential equation of the form (1), called an *asymptotically* *conservative symmetric hyperbolic equation.* An equation of the form (1) is symmetric hyperbolic if A^0 is positive definite and A^ν ($\nu = 1, \ldots, n$) are symmetric and is asymptotically conservative if B is antihermitian, i.e. $B^* = -B$. This terminology is appropriate because if $C = 0$ then (1) conserves energy. Since for large values of the parameter λ, $\lambda B\mathbf{u}$ is the dominant undifferentiated term, we say that (1) is asymptotically conservative. In the present paper we present techniques applicable to a more general class of equations of the form (1) which includes the asymptotically conservative symmetric hyperbolic case.

In general, the techniques needed to prove rigorously that the formal asymptotic solutions of the form (3) obtained in Lewis (1964, 1965) and this paper are actually asymptotic to the exact solution of a given problem are not known. Up to the present time the justification for calling the formal solution 'asymptotic' is that in problems where the asymptotic expansion as $\lambda \to \infty$ of the exact solution can be obtained explicitly, the formal solution and this expansion agree. However, in the case when (1) is symmetric hyperbolic and asymptotically conservative and the initial data for $\mathbf{u}$ is *oscillatory*, i.e.

$$\mathbf{u}(0, \mathbf{X}) = e^{i\lambda s_0(\mathbf{X})} \sum_{j=1}^{\infty} (i\lambda)^{-j} \mathbf{f}_j(\mathbf{X}), \tag{5}$$

it can be shown that the formal asymptotic solution (3) is asymptotic to the exact solution.† Data of the form (5) were introduced by Lax (1957) for a weakly dispersive equation.

The initial values for the characteristic equations and transport equations, which are required for their solution, must be obtained from the given data for $\mathbf{u}$, e.g. initial data and boundary data. For the oscillatory initial data, discussed in § 3, the required initial values can be derived directly from the given data for $\mathbf{u}$. On the other hand, several problems occur in which the initial values for the characteristic equations and transport equations cannot be found directly from the given data. For such a problem an *indirect method* is required for the determination of these quantities. This indirect method involves the consideration of a related problem called a *canonical problem*. A canonical problem is one with the same local properties as the given problem in the neighbourhood of the initial manifold or source function region. However, it is formulated in such a manner that it may be solved exactly. The asymptotic expansion as $\lambda \to \infty$ of this exact solution is then investigated in order to obtain the required initial values for the characteristic equations and transport equations. Several examples which require a canonical problem are treated in Lewis (1965). The concept of a canonical problem was introduced by Keller in his investigations of certain elliptic partial differential equations (1953 or 1962).

In § 5, we consider the initial boundary-value problem for the case when (1) is symmetric hyperbolic and asymptotically conservative. A linear homogeneous condition is imposed on the solution $\mathbf{u}$ of (1) at the boundary. Under certain restrictions, a formal asymptotic solution of the form (3) can be constructed by means of the principle of superposition which satisfies the given condition at the boundary. The problem of reflexion and refraction at an interface is considered in § 4. This differs from the boundary-value problem in that (1) and its solution are defined on both sides of a given boundary surface. Our construction is limited to the case when there is no total reflexion. In our investigation we found

† To be more specific, we show that the formal asymptotic solution is asymptotic to the exact solution with respect to a certain norm defined in § 3·2.

it convenient to treat this latter problem first and then consider the initial boundary-value problem as a special case.

It can be shown that the equations which govern electro-magnetic wave propagation in an isotropic nonconducting medium and in a plasma satisfy the various conditions necessary in order that the ray theory be applicable. In both of these examples the governing equations consist of Maxwell's equations and an equation of motion of an electron in the given medium. The parameter λ is an average value of a frequency associated with the medium.

2. The ray theory

2·1. *The formal expansion, derivation of dispersion relation and ray equations*

We consider a hyperbolic system of equations of the form

$$A^0 \frac{\partial \mathbf{u}}{\partial t} + \sum_{\nu=1}^{n} A^\nu \frac{\partial \mathbf{u}}{\partial x_\nu} + \lambda B\mathbf{u} + C\mathbf{u} = 0, \tag{6}$$

where the $m \times m$ matrices A^0; A^ν $(\nu = 1, \ldots, n)$; B; and C are smooth functions of t and $\mathbf{X} = (x_1, \ldots, x_n)$; $\mathbf{u}(t, \mathbf{X})$ is an m-dimensional column vector; and λ is a large positive parameter. The matrix A^0 is assumed to be nonsingular. Let us consider the formal asymptotic solution

$$\mathbf{u}(t, \mathbf{X}; \lambda) \sim e^{i\lambda s(t, \mathbf{X})} \sum_{j=0}^{\infty} (i\lambda)^{-j} \mathbf{z}_j(t, \mathbf{X}) \quad (\lambda \to \infty). \tag{7}$$

The function $s(t, \mathbf{X})$ is called the *phase function* and $\mathbf{z}_j(t, \mathbf{X})$ is called the jth order *amplitude function*. Inserting (7) into (6) and collecting powers of λ yields the recursive system of equations

$$G\mathbf{z}_{j+1} = -A^0 \frac{\partial \mathbf{z}_j}{\partial t} - \sum_{\nu=1}^{n} A^\nu \frac{\partial \mathbf{z}_j}{\partial x_\nu} - C\mathbf{z}_j \quad (j = -1, 0, \ldots). \tag{8}$$

Here

$$G(t, \mathbf{X}; \omega, \mathbf{K}) = \sum_{\nu=1}^{n} k_\nu A^\nu - iB - \omega A^0, \tag{9}$$

$$k_\nu = \frac{\partial s}{\partial x_\nu} \quad (\nu = 1, \ldots, n); \quad \omega = -\frac{\partial s}{\partial t},$$

and $\mathbf{z}_{-1}(t, \mathbf{X}) = 0$.

The existence nontrivial solutions (7) implies that

$$\det G(t, \mathbf{X}; \omega, \mathbf{K}) = 0. \tag{10}$$

The $m \times m$ matrix $G(t, \mathbf{X}; \omega, \mathbf{K})$ is called the *dispersion matrix* and (10) is called the *dispersion relation*. We assume that G has the properties:

(i) there exist p real distinct roots

$$\omega = h_j(t, \mathbf{X}, \mathbf{K}) \quad (j = 1, \ldots, p \leqslant m) \tag{11}$$

of (10),

(ii) associated with each root h_j there are q_j linearly independent vectors $\mathbf{r}^j_\mu(t, \mathbf{X}, \mathbf{K})$ such that

$$G\mathbf{r}^j_\mu = 0 \tag{12}$$

and, furthermore, $\sum_{j=1}^{p} q_j = m$ and the set of m vectors $\mathbf{r}^j_\mu$ $(j = 1, \ldots, p; \mu = 1, \ldots, q_j)$ is linearly independent,

(iii) the integers q_j are independent of t, $\mathbf{X}$, and $\mathbf{K}$. We call the vectors $\mathbf{r}^j_\mu$ ($\mu=1, \ldots, q_j$) the null vectors associated with the root $\omega = h_j$ and the integer q_j the multiplicity of h_j.

If (6) is an asymptotically conservative symmetric hyperbolic equation, i.e. if A^0 is positive definite, A^ν ($\nu=1, \ldots, n$) are symmetric; and B is antihermitian, then the dispersion matrix G is hermitian and it can be shown that G has properties (i) and (ii). Property (iii) is the only condition to be assumed. For this type of equation we may select the null vectors so that

$$[\mathbf{r}^i_\mu, A^0\mathbf{r}^j_\nu] = \delta_{ij}\,\delta_{\mu\nu}. \tag{13}$$

Let us consider a particular root $\omega = h(t, \mathbf{X}, \mathbf{K})$ of an arbitrary dispersion matrix G given by (9). Equation (11) may be regarded as a first order partial differential equation for the phase function $s(t, \mathbf{X})$ which can be solved by the method of characteristics. Its system of characteristic equations is

$$\frac{\mathrm{d}x_\nu}{\mathrm{d}t} = \frac{\partial h}{\partial k_\nu} = g_\nu, \quad \frac{\mathrm{d}k_\nu}{\mathrm{d}t} = -\frac{\partial h}{\partial x_\nu}, \quad \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\partial h}{\partial t} \quad (\nu=1, \ldots, n) \tag{14}$$

The first two equations in (14) define space-time curves $[t, \mathbf{X}(t)]$ which are called *rays*. We call the vector $\mathbf{G} = (g_1, \ldots, g_n)$ the *group velocity*. Along the rays $s(t, \mathbf{X})$ satisfies the equation

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sum_{\nu=1}^{n} k_\nu g_\nu - h. \tag{15}$$

In order to solve (14), (15), initial values for $\mathbf{X}$, $\mathbf{K}$, ω, and s are required. The phase function $s(t, \mathbf{X})$ is usually given on some initial manifold M of dimension $d \leqslant n$. From this we can derive initial values for $\mathbf{K}$ and ω at each $\mathbf{X}$. The method for obtaining these quantities is described in Lewis (1965).

2·2. *Derivation of the transport equations*

In this section we shall show that each amplitude function $\mathbf{z}_j(t, \mathbf{X})$ appearing in (7) can be found by solving a recursive system of ordinary differential equations. Our attention is restricted to a given root $\omega = h(t, \mathbf{X}, \mathbf{K})$ of (10) with multiplicity q. The discussion may be repeated for each distinct root.

We now introduce the adjoint G^* of the dispersion matrix G. It is easily seen from (9) that

$$G^*(t, \mathbf{X}, h, \mathbf{K}) = \sum_{\nu=1}^{n} k_\nu A^{\nu *} + iB^* - hA^{0*} \tag{16}$$

since $\mathbf{K}$ and $h(t, \mathbf{X}, \mathbf{K})$ are real-valued. Because the dimension of the null space of G is q, it follows that the null space of G^* has dimension q also. Therefore there exist q linearly independent vectors $\mathbf{p}_j(t, \mathbf{X}, \mathbf{K})$ ($j=1, \ldots, q$), such that

$$G^*\mathbf{p}_j = 0 \quad (j=1, \ldots, q). \tag{17}$$

By property (iii) of § 2·1, we may differentiate (12) with respect to k_ν to obtain

$$G\frac{\partial \mathbf{r}_j}{\partial k_\nu} + \frac{\partial G}{\partial k_\nu}\mathbf{r}_j = 0 \quad (j=1, \ldots, q). \tag{18}$$

Because of (17), the inner product of the left hand side of (18) and $\mathbf{p}_\mu$ yields

$$\left[\mathbf{p}_\mu, \frac{\partial G}{\partial k_\nu}\mathbf{r}_j\right] = 0 \quad (\mu, j=1, \ldots, q). \tag{19}$$

But from (9) and (14) we find that

$$\partial G/\partial k_\nu = A^\nu - g_\nu A^0. \tag{20}$$

Therefore, by inserting (20) into (19), we obtain

$$[\mathbf{p}_\mu, A^\nu \mathbf{r}_j] = g_\nu [\mathbf{p}_\mu, A^0 \mathbf{r}_j] \quad (\mu, j = 1, \ldots, q). \tag{21}$$

If we suppose that $A^0 = I$†, where I is the $m \times m$ identity matrix, (21) is simplified when we employ the following theorem.

THEOREM 2·1. If G satisfies conditions (i) and (ii) of § 2·1 and $A^0 = I$, then the vectors $\mathbf{p}_\mu$ $(\mu = 1, \ldots, q)$ can be selected so that

$$[\mathbf{p}_\mu, \mathbf{r}_j] = \delta_{\mu j} \quad (\mu, j = 1, \ldots, q). \tag{22}$$

The proof of this theorem is given in appendix I.

In the ensuing discussion we assume that $A^0 = I$. Then (21) may be rewritten as

$$[\mathbf{p}_\mu, A^\nu \mathbf{r}_j] = g_\nu \delta_{\mu j} \quad (\mu, j = 1, \ldots, q;\ \nu = 0, \ldots, \mu,\ (g_0 = 1)). \tag{23}$$

Equation (23) is called the *basic identity*. From (8) we have

$$-G\mathbf{z}_{j+1} = \frac{\partial \mathbf{z}_j}{\partial t} + \sum_{\nu=1}^{n} A^\nu \frac{\partial \mathbf{z}_j}{\partial x_\nu} + C\mathbf{z}_j \quad (j = -1, 0, \ldots). \tag{24}$$

Here $\mathbf{z}_{-1} = 0$.

From properties (i) and (ii) of § 2·1 it follows that there exists a basis of the m-dimensional vector space E^m consisting of the vectors $\mathbf{r}_1, \ldots, \mathbf{r}_m$ which satisfy

$$G\mathbf{r}_j = \begin{cases} 0 & (j = 1, \ldots, q) \\ (h_j - h)\,\mathbf{r}_j & (j = q+1, \ldots, m), \end{cases} \tag{25}$$

where $h_j - h \neq 0$ $(j = q+1, \ldots, m)$. Therefore the jth order amplitude function $\mathbf{z}_j(t, \mathbf{X})$ has the representation

$$\mathbf{z}_j(t, \mathbf{X}) = \sum_{i=1}^{m} \sigma_i^j(t, \mathbf{X}, \mathbf{K})\, \mathbf{r}_i(t, \mathbf{X}, \mathbf{K}). \tag{26}$$

We insert (26) into (24) and then take the inner product of (24) and $\mathbf{p}_\mu$ $(\mu = 1, \ldots, q)$. With the aid of the basic identity (23), we obtain the equation

$$\frac{\partial \sigma_\mu^j}{\partial t} + \sum_{\nu=1}^{n} g_\nu \frac{\partial \sigma_\mu^j}{\partial x_\nu} + \sum_{i=1}^{q} \tau_{\mu i} \sigma_i^j = \gamma_\mu^j \quad (\mu = 1, \ldots, q), \tag{27}$$

where

$$\tau_{\mu i} = \left[\mathbf{p}_\mu, \frac{\partial \mathbf{r}_i}{\partial t} + \sum_{\nu=1}^{n} A^\nu \frac{\partial \mathbf{r}_i}{\partial x_\nu} + C\mathbf{r}_i\right] \tag{28}$$

and

$$\gamma_\mu^j = -\sum_{i=q+1}^{m} \left\{[\mathbf{p}_\mu, \mathbf{r}_i] \frac{\partial \sigma_i^j}{\partial t} + \sum_{\nu=1}^{n} [\mathbf{p}_\mu, A^\nu \mathbf{r}_j] \frac{\partial \sigma_i^j}{\partial x_\nu} + \tau_{\mu i} \sigma_i^j\right\}. \tag{29}$$

† There is no loss of generality if we assume $A^0 = I$ because multiplication of (6) by $(A^0)^{-1}$ on the left gives

$$I\frac{\partial \mathbf{u}}{\partial t} + \sum_{\nu=1}^{n} \hat{A}^\nu \frac{\partial \mathbf{u}}{\partial x_\nu} + \lambda \hat{B}\mathbf{u} + \hat{C}\mathbf{u} = 0,$$

where $\hat{A}^\nu = (A^0)^{-1} A^\nu$, $\hat{B} = (A^0)^{-1} B$, $\hat{C} = (A^0)^{-1} C$. If (6) is hyperbolic and if its dispersion matrix G has properties (i to iii) of § 2·1, then the above equation is also hyperbolic and its associated dispersion matrix $\hat{G} = (A^0)^{-1} G$ has properties (i to iii) also. We may now treat this equation instead of (6) in the manner described above.

A consequence of (14) is that
$$\frac{\partial \sigma^j_\mu}{\partial t} + \sum_{\nu=1}^{n} g_\nu \frac{\partial \sigma^j_\mu}{\partial x_\nu} = \frac{\mathrm{d}\sigma^j_\mu}{\mathrm{d}t}.$$

Thus (27) becomes the inhomogeneous system of ordinary differential equations

$$\frac{\mathrm{d}\sigma^j_\mu}{dt} + \sum_{i=1}^{q} \tau_{\mu i}\sigma^j_i = \gamma^j_\mu \quad (\mu = 1, \ldots, q). \tag{30}$$

This system is called the *jth order transport equations*. We now show that the functions γ^j_μ $(\mu = 1, \ldots, q)$, can be found from the $(j-1)$st order amplitude function $\mathbf{z}_{j-1}$. Replacing j by $j-1$ in (24) and then substituting (26) into (24) results in

$$\sum_{i=q+1}^{w} \sigma^j_i G\mathbf{r}_i = -\left(\frac{\partial \mathbf{z}_{j-1}}{\partial t} + \sum_{\nu=1}^{n} A^\nu \frac{\partial \mathbf{z}_{j-1}}{\partial x_\nu} + C\mathbf{z}_{j-1}\right). \tag{31}$$

By (25), we obtain from (31) the algebraic system of equations

$$\sum_{i=q+1}^{m} \sigma^j_i (h_i - h)\, \mathbf{r}_i = -\left(\frac{\partial \mathbf{z}_{j-1}}{\partial t} + \sum_{\nu=1}^{n} A^\nu \frac{\partial \mathbf{z}_{j-1}}{\partial x_\nu} + C\mathbf{z}_{j-1}\right). \tag{32}$$

Since the right hand side of (32) is orthogonal to the null space of G^*, the unknown functions σ^j_i $(i = q+1, \ldots, m)$, in (32) can be uniquely determined. Inserting σ^j_i into (29) we obtain γ^j_μ. We observe that for $j = 0$, the right-hand side of (32) vanishes and $\sigma^0_i = 0$ $(i = q+1, \ldots, m)$. From (29) we find that $\gamma^0_\mu = 0$ $(\mu = 1, \ldots, q)$. Hence the zero order transport equations are homogeneous ordinary differential equations.

If (6) is an asymptotically conservative symmetric hyperbolic equation, the dispersion matrix G is hermitian and we may set $\mathbf{p}_\mu = \mathbf{r}_\mu$. For this type of equation theorem 2·1 is not necessary since (23) follows directly from (13). We note that in this instance the matrix A^0 need not be the identity matrix. The jth order transport equations are again given by (30) where τ_{ij} are given by (13) with $\mathbf{p}_\mu$ replaced by $\mathbf{r}_\mu$ and $\partial\mathbf{r}_i/\partial t$ replaced by $A^0(\partial\mathbf{r}_i/\partial t)$.

2·3. *Solution of the transport equations*

In general, the transport equations (30) form a system of linear first order ordinary differential equations with variable coefficients which cannot be solved explicitly. However, under certain restrictions, the transport equations reduce to a system of equations which can be solved exactly by standard methods. There are three specific cases of interest for which the reduction can be accomplished. They are:

Case 1. The matrices A^ν $(\nu = 1, \ldots, \mu,)$ B and C are constant.

Case 2. The multiplicity of a given root is one.

Case 3. The multiplicity of a given root is two and

$$\tau_{11} = \tau_{22} \quad \text{and} \quad \tau_{12} = -\tau_{21}, \quad \text{where } \tau_{ij} \text{ are given by (28).}$$

The last case is important because it frequently occurs in problems of physical interest such as electromagnetic wave propagation.

The verification that the above-mentioned cases can be solved explicitly is given in §§ 3·6, 3·8, and 3·9 of Lewis (1965). There the solutions are even exhibited. In the introduction to this paper we mentioned that certain problems exist for which the initial data for the transport equations cannot be found directly. For these problems a canonical problem is required. The treatment of these types of problems is also discussed in Lewis (1965) in §§3·10 and 3·11. The canonical problems required for the transport equations derived in the preceding section are given in Granoff & Lewis (1966).

3. Oscillatory initial data

3·1. *Initial values for the ray equations, phase equation, and transport equation*

In this section we derive the initial values of the ray equations, phase equation, and transport equations from a particular initial value problem of (6). At $t = 0$ we suppose that the initial data for $\mathbf{u}$ is

$$\mathbf{u}(0, \mathbf{X}) = e^{i\lambda s_0(\mathbf{X})} \sum_{\mu=0}^{\infty} (i\lambda)^{-\mu} \mathbf{f}_\mu(\mathbf{X}). \tag{33}$$

Initial data of this form is called *oscillatory initial data*. It was introduced by Lax (1957). We shall derive all the required initial values directly from (33).

For each distinct root $\omega = h_j(t, X, K)$ of the dispersion relation (10) we can construct a formal solution $\mathbf{u}_j(t, \mathbf{X})$ of the form (7). The asymptotic solution of (6) is then given as the sum of these solutions, i.e.

$$\mathbf{u}(t, \mathbf{X}) \sim \sum_{j=1}^{p} \mathbf{u}_j(t, \mathbf{X}), \tag{34}$$

where

$$\mathbf{u}_j(t, \mathbf{X}) = e^{i\lambda s_j(t, \mathbf{X})} \sum_{\mu=0}^{\infty} (i\lambda)^{-\mu} \mathbf{z}_{\mu j}(t, \mathbf{X}). \tag{35}$$

From (33) we obtain

$$\sum_{\mu=0}^{\infty} (i\lambda)^{-\mu} e^{i\lambda s_0(\mathbf{X})} \mathbf{f}_\mu(\mathbf{X}) = \sum_{\mu=0}^{\infty} (i\lambda)^{-\mu} \sum_{j=1}^{p} e^{i\lambda s_j(0, \mathbf{X})} \mathbf{z}_{\mu j}(0, \mathbf{X}). \tag{36}$$

In order that (36) be valid for all large values of λ, we require

$$s_j(0, X) = s_0(X) \quad (j = 1, \ldots, p). \tag{37}$$

Then

$$\sum_{j=1}^{p} \mathbf{z}_{\mu j}(0, \mathbf{X}) = \mathbf{f}_\mu(\mathbf{X}) \quad (\mu = 0, 1, \ldots). \tag{38}$$

We now write $\mathbf{z}_{\mu j}$ as

$$\mathbf{z}_{\mu j} = \sum_{i=1}^{q_j} \sigma_{\mu j i} \mathbf{r}_i^j + \mathbf{v}_{\mu j}, \tag{39}$$

where $\mathbf{r}_i^j$ are the null vectors associated with $\omega = h_j$ and $\mathbf{v}_{\mu j}$ is a vector that can be found from $\mathbf{z}_{\mu-1, j}$ by (31). When $\mu = 0$, $\mathbf{v}_{0j} = 0$. Substitution of (39) into (38) yields

$$\sum_{j=1}^{p} \sum_{i=1}^{q_j} \sigma_{\mu j i} \mathbf{r}_i^j = \mathbf{f}_\mu - \sum_{j=1}^{p} \mathbf{v}_{\mu j}. \tag{40}$$

Since $\mathbf{r}_i^j$ are linearly independent, the quantities $\sigma_{\mu j i}$ can be uniquely determined. These quantities are the initial values for the μth order transport equations.

Equation (37) gives the initial value for all the phase functions $s_j(t, X)$. The initial value for the wave vector $\mathbf{K}$ is given by

$$k_\nu = \frac{\partial s_0(\mathbf{X})}{\partial x_\nu} \quad (\nu = 1, \ldots, n). \tag{41}$$

Thus we have derived all the required initial values for the ray equations, phase equation, and transport equations.

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3·2. *Proof of the asymptotic nature of the formal solution for the asymptotically conservative symmetric hyperbolic equation with oscillatory initial data*

We noted in the introduction to this paper that, in general, we do not know how to prove rigorously the asymptotic nature of the formal solution (7). In the following discussion we present the exceptional case for which precise statements can be made concerning the formal solution.

Let us consider the asymptotically conservative symmetric hyperbolic equation

$$\frac{\partial \mathbf{u}}{\partial t}+\sum_{\nu=1}^{n} A^{\nu}\frac{\partial \mathbf{u}}{\partial x_{\nu}}+\lambda B\mathbf{u}+C\mathbf{u}=0. \tag{42}$$

Here A^{ν} $(\nu=1,\ldots,n)$, are smooth symmetric matrix functions of $(t,\mathbf{X})$, B is a smooth anti-hermitian matrix function of $(t,\mathbf{X})$, and C is a smooth arbitrary matrix. We prescribe initial data of the form (33) for (42). In addition, it is assumed that (33) has compact support. The formal asymptotic solution is given by (34).

We delete all the terms in (34) after the ρth term to obtain

$$\mathbf{U}_{\rho}=\sum_{\mu=0}^{\rho}(\mathrm{i}\lambda)^{-\mu}\sum_{j=1}^{p}\mathrm{e}^{\mathrm{i}\lambda s_j(t,\,\mathbf{X})}\mathbf{z}_{\mu j}(t,\mathbf{X}). \tag{43}$$

Let us suppose that $\mathbf{U}$ is the exact solution of (42) and (33). Our goal is to estimate the difference $\boldsymbol{\theta}_{\rho}=\mathbf{U}_{\rho}-\mathbf{U}$. Since $s_j(t,\mathbf{X})$ and $\mathbf{z}_{\mu j}(t,\mathbf{X})$ satisfy the phase and transport equations respectively, insertion of $\boldsymbol{\theta}_{\rho}$ into (42) results in

$$\frac{\partial \boldsymbol{\theta}_{\rho}}{\partial t}+\sum_{\nu=1}^{n}A^{\nu}\frac{\partial \boldsymbol{\theta}_{\rho}}{\partial x_{\nu}}+\lambda B\boldsymbol{\theta}_{\rho}+C\boldsymbol{\theta}_{\rho}=(\mathrm{i}\lambda)^{-\rho}\sum_{j=1}^{p}\mathrm{e}^{\mathrm{i}\lambda s_j}\boldsymbol{\psi}_j, \tag{44}$$

where
$$\boldsymbol{\psi}_j=\frac{\partial \mathbf{z}_{\rho j}}{\partial t}+\sum_{\nu=1}^{n}A^{\nu}\frac{\partial \mathbf{z}_{\rho j}}{\partial x_{\nu}}+C\mathbf{z}_{\rho j}.$$

It is clear that $\boldsymbol{\psi}_j$ is independent of λ. From (33) we find that

$$\boldsymbol{\theta}_{\rho}(0,\mathbf{X};\lambda)=(\mathrm{i}\lambda)^{-\rho-1}\,\mathrm{e}^{\mathrm{i}\lambda s_0(\mathbf{X})}\sum_{\mu=0}^{\infty}(\mathrm{i}\lambda)^{-\mu}\mathbf{f}_{\rho+1+\mu}(\mathbf{X}). \tag{45}$$

Let us represent the solution of (44), (45) as

$$\boldsymbol{\theta}_{\rho}(t,\mathbf{X};\lambda)=\boldsymbol{\theta}_{\rho}^{1}(t,\mathbf{X};\lambda)+\boldsymbol{\theta}_{\rho}^{2}(t,\mathbf{X};\lambda), \tag{46}$$

where $\boldsymbol{\theta}_{\rho}^{1}$ satisfies (42) and (45) and $\boldsymbol{\theta}_{\rho}^{2}$ satisfies (44) and

$$\boldsymbol{\theta}_{\rho}^{2}(0,\mathbf{X};\lambda)=0. \tag{47}$$

The solution of (44), (47) can be found by Duhamel's principle (Courant & Hilbert 1962). Application of this principle yields the solution

$$\boldsymbol{\theta}_{\rho}^{2}(t,\mathbf{X};\lambda)=(\mathrm{i}\lambda)^{-\rho}\int_{0}^{t}\boldsymbol{\phi}(t,\mathbf{X};\tau;\lambda)\,\mathrm{d}\tau, \tag{48}$$

where $\boldsymbol{\phi}(t,\mathbf{X};\tau;\lambda)$ satisfies (42) for $t>\tau$ and

$$\boldsymbol{\phi}(\tau,\mathbf{X};\tau;\lambda)=\sum_{j=1}^{p}\mathrm{e}^{\mathrm{i}\lambda s_j}\boldsymbol{\psi}_j. \tag{49}$$

Since (33) has compact support, it can be shown that $\boldsymbol{\phi}(\tau,\mathbf{X};\tau;\lambda)$ has compact support also.

Under the conditions given above, the following theorem is valid.

THEOREM 3·1 $$\|\boldsymbol{\theta}_\rho(t)\| = \|\mathbf{U}_\rho - \mathbf{U}\| = O(\lambda^{-\rho+1}) \quad (0 \leqslant t < t'), \tag{50}$$

where t' is the time of the first caustic,† i.e.

$$\lambda^{\rho-1}\|\mathbf{U}_\rho - \mathbf{U}\| \to 0 \quad \text{as} \quad \lambda \to \infty \quad \text{for} \quad 0 \leqslant t < t'.$$

Proof. The proof requires the two following lemmas.

LEMMA 3·1.
$$\|\boldsymbol{\phi}(t, \tau; \lambda)\|^2 = \int_{-\infty}^{\infty} [\boldsymbol{\phi}(t, \mathbf{X}, \tau, \lambda), \boldsymbol{\phi}(t, \mathbf{X}, \tau, \lambda)]\, \mathrm{d}\mathbf{X} \leqslant \alpha^2(t, \tau), \tag{51}$$

i.e. the norm of $\boldsymbol{\phi}$ is bounded independently of the parameter λ. At a caustic $\alpha(t, \tau)$ becomes infinite.

Proof of lemma 3·1. If we set $\hat{\boldsymbol{\phi}} = \mathrm{e}^{-\mu t}\boldsymbol{\phi}$ where μ is a positive constant, then, by direct substitution, we see that $\boldsymbol{\phi}$ satisfies

$$\frac{\partial \hat{\boldsymbol{\phi}}}{\partial t} + \sum_{\nu=1}^{n} A^\nu \frac{\partial \hat{\boldsymbol{\phi}}}{\partial x_\nu} + \lambda B \hat{\boldsymbol{\phi}} + (C + \mu I)\, \hat{\boldsymbol{\phi}} = 0. \tag{52}$$

The constant μ will be determined shortly. In Courant & Hilbert (1962) it is shown that (52) implies

$$\frac{\partial[\hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\phi}}]}{\partial t} + \sum_{\nu=1}^{n} \frac{\partial[\hat{\boldsymbol{\phi}}, A^\nu \hat{\boldsymbol{\phi}}]}{\partial x_\nu} + [\hat{\boldsymbol{\phi}}, (D + \lambda B + \lambda B^*)\, \hat{\boldsymbol{\phi}}] = 0, \tag{53}$$

where $$D = \mu I - \sum_{\nu=1}^{n} \frac{\partial A^\nu}{\partial x_\nu} + C + C^*.$$

Since B is antihermitian, $B + B^* = 0$. For a given region of space-time, μ can be selected sufficiently large so that the matrix D is positive definite in the given region. It is clear that μ depends in no way on the parameter λ. The energy inequality (Courant & Hilbert 1962) for $\boldsymbol{\phi}$ is

$$\int_{D_t} [\hat{\boldsymbol{\phi}}(t, \mathbf{X}, \tau, \lambda), \hat{\boldsymbol{\phi}}(t, \mathbf{X}, \tau, \lambda)]\, \mathrm{d}\mathbf{X} \leqslant \int_{D_\tau} [\hat{\boldsymbol{\phi}}(\tau, \mathbf{X}, \tau, \lambda), \hat{\boldsymbol{\phi}}(\tau, \mathbf{X}, \tau, \lambda)]\, \mathrm{d}\mathbf{X}. \tag{54}$$

Here $t > \tau$ and D_t, D_τ are domains of integration defined by a backward drawn 'conoid of dependence'. For a detailed description of D_t and D_τ see Courant & Hilbert (1962). Since $\hat{\boldsymbol{\phi}}(\tau, \mathbf{X}, t, \lambda)$ has compact support, it can be shown that the domains D_t and D_τ can be replaced by the entire $\mathbf{X}$-space and (54) will still hold. Hence

$$\|\hat{\boldsymbol{\phi}}(t, \tau, \lambda)\|^2 = \int_{-\infty}^{\infty} [\hat{\boldsymbol{\phi}}(t, \mathbf{X}, \tau, \lambda), \hat{\boldsymbol{\phi}}(t, \mathbf{X}, \tau, \lambda)]\, \mathrm{d}\mathbf{X}$$

$$\leqslant \int_{-\infty}^{\infty} [\hat{\boldsymbol{\phi}}(\tau, \mathbf{X}, \tau, \lambda), \hat{\boldsymbol{\phi}}(\tau, \mathbf{X}, \tau, \lambda)]\, \mathrm{d}\mathbf{X} = \|\hat{\boldsymbol{\phi}}(\tau, \tau, \lambda)\|^2. \tag{55}$$

From the definition of $\hat{\boldsymbol{\phi}}$ and (55) we obtain

$$\|\hat{\boldsymbol{\phi}}(t, \tau, \lambda)\|^2 \leqslant \mathrm{e}^{2\mu(t-\tau)}\|\boldsymbol{\phi}(\tau, \tau, \lambda)\|^2. \tag{56}$$

Insertion of (49) into $\|\boldsymbol{\phi}(\tau, \tau, \lambda)\|^2$ yields the inequality

$$\|\boldsymbol{\phi}(\tau, \tau, \lambda)\|^2 = \sum_{j,k=1}^{p} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda(s_j - s_k)}[\boldsymbol{\psi}_j, \boldsymbol{\psi}_k]\, \mathrm{d}\mathbf{X} \leqslant \sum_{j,k=1}^{p} \int_{-\infty}^{\infty} |[\boldsymbol{\psi}_j, \boldsymbol{\psi}_k]|\, \mathrm{d}\mathbf{X}. \tag{57}$$

† For a discussion of caustics see Lewis (1965).

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If we set
$$e^{2\mu(t-\tau)} \sum_{j,k=1}^{p} \int_{-\infty}^{\infty} |[\boldsymbol{\psi}_j, \boldsymbol{\psi}_k]| \, d\mathbf{X} = \alpha^2(t,\tau), \tag{58}$$
then (56) and (57) imply (51). Since $\boldsymbol{\psi}_j$ is infinite at a caustic, so is $\alpha(t,\tau)$.

LEMMA 3·2.
$$\|\boldsymbol{\theta}^1_\rho(t)\| = \left\{\int_{-\infty}^{\infty} [\boldsymbol{\theta}^1_\rho(t,\mathbf{X},\lambda), \boldsymbol{\theta}^1_\rho(t,\mathbf{X},\lambda)] \, d\mathbf{X}\right\}^{\frac{1}{2}} = O(\lambda^{-\rho}). \tag{59}$$

Proof of lemma 3·2. In a manner described in the preceding theorem we may derive the inequality
$$\|\boldsymbol{\theta}^1_\rho(t,\lambda)\|^2 \leqslant e^{2\mu t} \|\boldsymbol{\theta}^1_\rho(0,\lambda)\|^2. \tag{60}$$
Insertion of (45) into (60) yields
$$\|\boldsymbol{\theta}^1_\rho(t,\lambda)\|^2 \leqslant \lambda^{-2\rho-2} e^{2\mu t} \int_{-\infty}^{\infty} \left[\sum_{\mu=0}^{\infty} (i\lambda)^{-\mu} \mathbf{f}_{\rho+1+\mu}(\mathbf{X}), \sum_{\mu=0}^{\infty} (i\lambda)^{-\mu} \mathbf{f}_{\rho+1+\mu}(\mathbf{X})\right] d\mathbf{X} \tag{61}$$
$$= \lambda^{-2\rho-2}\beta^2(t) + O(\lambda^{-2\rho-3}).$$
The statement of the lemma follows immediately from (61).

Proof of theorem 3·1. From (48) we find that
$$\|\boldsymbol{\theta}^2_\rho(t,\lambda)\|^2 = \lambda^{-2\rho} \int_{-\infty}^{\infty} \left[\int_0^t \boldsymbol{\phi}(t,\mathbf{X};\tau;\lambda) \, d\tau, \int_0^t \boldsymbol{\phi}(t,\mathbf{X};\tau;\lambda) \, d\tau\right] d\mathbf{X}. \tag{62}$$
Since $\boldsymbol{\phi}$ is an m-component column vector
$$\left[\int_0^t \boldsymbol{\phi}(t,\mathbf{X};\tau;\lambda) \, d\tau, \int_0^t \boldsymbol{\phi}(t,\mathbf{X};\tau;\lambda) \, d\tau\right] = \sum_{j=1}^{m} \left|\int_0^t \phi_j(t,\mathbf{X};\tau;\lambda) \, d\tau\right|^2$$
$$\leqslant \sum_{j=1}^{m} \left[\int_0^t |\phi_j(t,\mathbf{X};\tau;\lambda)| \, d\tau\right]^2. \tag{63}$$
By Schwarz's inequality,
$$\left[\int_0^t |\phi_j(t,\mathbf{X};\tau;\lambda)| \, d\tau\right]^2 \leqslant t \int_0^t |\phi_j(t,\mathbf{X};\tau;\lambda)|^2 \, d\tau. \tag{64}$$
From (63) and (64) we obtain
$$\left[\int_0^t \boldsymbol{\phi}(t,\mathbf{X};\tau;\lambda) d\tau, \int_0^t \boldsymbol{\phi}(t,\mathbf{X},\tau,\lambda) \, d\tau\right]$$
$$\leqslant t \int_0^t \sum_{j=1}^{m} [\phi_j(t,\mathbf{X};\tau;\lambda)]^2 \, d\tau = t \int_0^t [\boldsymbol{\phi}(t,\mathbf{X},\tau,\lambda), \boldsymbol{\phi}(t,\mathbf{X},\tau,\lambda)] \, d\tau. \tag{65}$$
From (62) and (65) it follows that
$$\|\boldsymbol{\theta}^2_\rho(t,\lambda)\|^2 \leqslant \lambda^{-2\rho} t \int_0^t \|\boldsymbol{\phi}(t,\tau,\lambda)\|^2 \, d\tau. \tag{66}$$
With the aid of lemma 3·1 we obtain from (66) the inequality
$$\|\boldsymbol{\theta}^2_\rho(t,\lambda)\|^2 \leqslant \lambda^{-2\rho} t \int_0^t \alpha^2(t,\tau) \, d\tau \quad \text{for} \quad 0 \leqslant t \leqslant t', \tag{67}$$
where t' is the time of the first caustic. Hence
$$\|\boldsymbol{\theta}^2_\rho(t,\lambda)\| = O(\lambda^{-\rho+1}). \tag{68}$$
Now, by (68) and lemma 3·2,
$$\|\boldsymbol{\theta}_\rho(t,\lambda)\| = \|\boldsymbol{\theta}^1_\rho(t,\lambda) + \boldsymbol{\theta}^2_\rho(t,\lambda)\| \leqslant \|\boldsymbol{\theta}^1_\rho(t,\lambda)\| + \|\boldsymbol{\theta}^2_\rho(t,\lambda)\|$$
$$= O(\lambda^{-\rho}) + O(\lambda^{-\rho+1}). \tag{69}$$
Therefore $\|\boldsymbol{\theta}_\rho(t,\lambda)\| = O(\lambda^{-\rho+1})$.

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Thus, with respect to the norm $\| \|$, the formal asymptotic solution (34) is asymptotic to the exact solution $\mathbf{U}$.

Without further restrictions on the matrix B, we have not been able to prove stronger point-wise results. If B is a constant antihermitian matrix, we can obtain point-wise estimates by means of Sobolev's lemma (Courant & Hilbert 1962). In general, Sobolev's lemma cannot be used because the λ dependence cannot be eliminated from the energy inequalities of higher order.

Maslov (1963) has proved a theorem similar to theorem 3·1 for Dirac's equation.

4. Reflexion and refraction at an interface

4·1. *Introduction*

In this chapter we shall investigate the solution of the initial value problem for the asymptotically conservative symmetric hyperbolic equation

$$\left.\begin{aligned} A^0 \frac{\partial \mathbf{u}}{\partial t} + \sum_{\nu=1}^{\mu} A^\nu \frac{\partial \mathbf{u}}{\partial x_\nu} + \lambda B\mathbf{u} + C\mathbf{u} = \mathbf{f}(t, \mathbf{X}, \lambda), \\ \mathbf{u}(0, \mathbf{X}) = \mathbf{v}(\mathbf{X}; \lambda), \end{aligned}\right\} \tag{70}$$

for which some matrix coefficients have a jump discontinuity across a smooth surface Ω: $\psi(\mathbf{X}) = 0$. To be more specific we assume that the matrices $A^\nu(t, \mathbf{X})$ $(\nu = 1, \ldots, \mu)$ are smooth for all $\mathbf{X}$, the matrices $A^0(t, \mathbf{X})$, $B(t, \mathbf{X})$, and $C(t, \mathbf{X})$ have a jump discontinuity across the surface Ω but are otherwise smooth, and that the functions $\mathbf{v}(\mathbf{X}; \lambda)$ and $\mathbf{f}(t, \mathbf{X}; \lambda)$ vanish in a neighbourhood of Ω.

The interface condition we shall impose on the solution $\mathbf{u}$ of (70) is given by the equation

$$A[\mathbf{u}(t, \mathbf{X})] = 0 \quad (\mathbf{X} \in \Omega). \tag{71}$$

Here $A = \sum_{\nu=1}^{\mu} A^\nu \eta_\nu$, $\mathbf{N} = \{\eta_1, \ldots, \eta_\mu\}$ is the unit normal to Ω, and $[\mathbf{u}]$ is the limiting value of the difference between $\mathbf{u}$ in the two regions of space separated by Ω. This condition states that the vector $[\mathbf{u}]$ belongs to the null space of the matrix A. If we suppose that the orthogonal complement of the null space is spanned by $\mathbf{Q}_1, \ldots, \mathbf{Q}_\rho$, where ρ is the rank of A, then a statement equivalent to (71) is

$$[\mathbf{Q}_\nu, [\mathbf{u}]] = 0; \quad \nu = 1, \ldots, \rho; \quad \mathbf{X} \in \Omega. \tag{72}$$

In appendix II it is shown that (71) is a natural interface condition to impose. An example of this type of condition is to be found in electromagnetic theory. In Jones (1964) it is shown that sufficient conditions to determine the electric and magnetic fields are $\mathbf{N} \wedge [\mathbf{E}] = 0$ and $\mathbf{N} \wedge [\mathbf{H}] = 0$. In Maxwell's equations the form of the matrix A is

$$A = \begin{pmatrix} 0 & -(\mathrm{N}) \\ (\mathrm{N}) & 0 \end{pmatrix}, \quad (\mathrm{N}) = \begin{pmatrix} 0 & -\eta_3 & -\eta_2 \\ \eta_3 & 0 & -\eta_1 \\ \eta_2 & \eta_1 & 0 \end{pmatrix}, \quad (\mathrm{N})\mathbf{V} = \mathbf{N} \wedge \mathbf{V}.$$

Therefore the interface condition is

$$\begin{pmatrix} 0 & -(\mathrm{N}) \\ (\mathrm{N}) & 0 \end{pmatrix} \begin{pmatrix} [\mathbf{E}] \\ [\mathbf{H}] \end{pmatrix} = 0$$

which implies the usual conditions.

The problem we are about to treat may be described in the following manner. We suppose that families of rays emanating from either the initial manifold or source region, or other interfaces or boundaries intersect the surface Ω with increasing t. With each family of rays we have an associated *wave*, i.e. a formal solution of (1) of the form (7). Our procedure will be to construct additional rays emanating from the interface Ω into the regions $\mathscr{D}_1$: $\psi(\mathbf{X}) > 0$ and $\mathscr{D}_2$: $\psi(\mathbf{X}) < 0$. Then, for each 'incident' wave, we find additional waves associated with these rays such that the sum of the 'incident' wave and constructed waves satisfies the interface condition (71). If the incident rays are in $\mathscr{D}_1$, then the additional rays in $\mathscr{D}_1$ are called reflected rays and the additional rays in $\mathscr{D}_2$ are called refracted rays.

The rays, phase function, and amplitude functions in (7) must satisfy the ray equations, phase equation, and transport equations respectively. They are uniquely determined once initial values for the ray, phase, and transport equations are given. The object of the following sections is to derive these initial values on the surface Ω in order to complete the construction of the reflected and refracted waves.

Given an incident wave, we see that the phase functions s appearing in all its reflected and refracted waves must be equal to the phase function s_0 of the incident wave on the surface Ω. This is necessary because the interface condition (72) must be satisfied for all large values of the parameter λ. From this property we can obtain the initial values for the reflected and refracted ray equations and phase equations on Ω. The application of the interface condition then leads to a linear algebraic system of equations for the initial values of the transport equations on Ω. Under certain restrictions we shall demonstrate that this algebraic system has a unique solution. Since the interface condition is linear the sum of all incident waves and their corresponding reflected and refracted waves will satisfy it.

4·2. *Initial values for the ray and phase equations on the interface*

In each of the regions $\mathscr{D}_1$ and $\mathscr{D}_2$ there is a dispersion relation (10). Corresponding to each dispersion relation there are ray equations (14) and a phase equation (15). We have indicated in the preceding section that the phase of the reflected and refracted waves must be equal to the phase s_0 of the incident wave on Ω. Therefore the initial value for the phase equation (15) in both regions $\mathscr{D}_1$ and $\mathscr{D}_2$ is s_0. It is obvious that the initial values for the $\mathbf{X}$ ray equations are the points of intersection of the incident rays and the surface Ω. Initial values for the $\mathbf{K}$ and ω equations appearing in (14) remain to be found. Since

$$s(t,\mathbf{X}) = s_0(t,\mathbf{X}) \quad (\mathbf{X}\in\Omega) \tag{73}$$

for the phase s of every reflected and refracted wave, differentiation of (73) with respect to t yields

$$\omega(t,\mathbf{X}) = \omega_0(t,\mathbf{X}) \quad (\mathbf{X}\in\Omega). \tag{74}$$

Let us express the surface Ω parametrically by the equations

$$x_\nu = x_\nu(\xi_1, \ldots, \xi_{n-1}) \quad (\nu = 1, \ldots, n). \tag{75}$$

Then differentiation of (73) with respect to ξ_μ $(\mu = 1, \ldots, n-1)$ results in

$$\sum_{\nu=1}^{n} \frac{\partial s}{\partial x_\nu}\frac{\partial x_\nu}{\partial \xi_\mu} = \sum_{\nu=1}^{n} \frac{\partial s_0}{\partial x_\nu}\frac{\partial x_\nu}{\partial \xi_\mu} \quad (\mathbf{X}\in\Omega, \mu = 1, \ldots, n-1). \tag{76}$$

We may rewrite (76) as
$$(\mathbf{K}-\mathbf{K}_0)\,\mathbf{J}_\mu = 0 \quad (\mu = 1, \ldots, n-1), \tag{77}$$

where
$$\mathbf{K} = \left(\frac{\partial s}{\partial x_\nu}\right), \quad \mathbf{K}_0 = \left(\frac{\partial s_0}{\partial x_\nu}\right) \quad \text{and} \quad \mathbf{J}_\mu = \left(\frac{\partial x_\nu}{\partial \xi_\mu}\right).$$

The vectors $\mathbf{J}_\mu$ $(\mu = 1, \ldots, n-1)$, are linearly independent since Ω is a smooth surface and each $\mathbf{J}_\mu$ is orthogonal to $\mathbf{N}$, the unit normal to Ω. Therefore (77) implies that

$$\mathbf{K} = \alpha\mathbf{N} + \mathbf{K}_0. \tag{78}$$

In order to find the initial values of $\mathbf{K}$ in $\mathscr{D}_1$ or $\mathscr{D}_2$ we must determine the values of the scalar α for $\mathbf{X} \in \Omega$ as the point $\mathbf{X}$ is approached from $\mathscr{D}_1$ or $\mathscr{D}_2$. Let us consider region $\mathscr{D}_1$. Let $h(t, \mathbf{X}, \mathbf{K})$ be a root of the dispersion relation in $\mathscr{D}_1$. Then (74) and (78) imply that

$$h(t, \mathbf{X}, \alpha\mathbf{N} + \mathbf{K}_0) = \omega_0 \quad (\mathbf{X} \in \Omega). \tag{79}$$

Now (79) is to be solved for the scalar α. There may be zero, one, or several real distinct solutions for each h†. In any case, it can be shown that there is a finite number of solutions α for each h. If there are no real solutions α then the root h does not generate any rays, i.e. no rays which are solution of the ray equations for that h appears. For every real distinct solution α of (79) there is a ray because for every real distinct α we have an initial value of the vector $\mathbf{K}$ given by (78). This procedure is repeated for every distinct root of the dispersion relation in $\mathscr{D}_1$. In this manner we derive initial values for the vector $\mathbf{K}$ for each root of the dispersion relative in $\mathscr{D}_1$. We carry out the same procedure to obtain initial values of $\mathbf{K}$ in region $\mathscr{D}_2$.

4·3. *Reflexion and refraction at the interface*

Let us suppose that the incident ray originates in region $\mathscr{D}_1$. Then the reflected rays are those rays which are constructed from the roots of the dispersion relation in $\mathscr{D}_1$ on the interface Ω and proceed into region $\mathscr{D}_1$ with increasing values of t. We require increasing values of t in order that the reflected waves do not disturb the initial condition for $\mathbf{u}$‡. The reflected rays satisfy the inequality

$$\mathbf{N} \cdot \dot{\mathbf{X}} = \sum_{\nu=1}^{n} \eta_\nu \dot{x}_\nu \geqslant 0 \quad \text{on} \quad \Omega, \tag{80}$$

where $\mathbf{N}$ is the unit normal to Ω which is directed into $\mathscr{D}_1$. From the basic identity (23) we may obtain the formula

$$\dot{x}_\nu = [\mathbf{r}, A^\nu \mathbf{r}], \tag{81}$$

where $\mathbf{r}$ is any null vector corresponding to a root $\omega = h$. Substitution of (81) into (80) yields

$$[\mathbf{r}, A\mathbf{r}] \geqslant 0 \quad \text{on } \Omega. \tag{82}$$

Here A is given by (71).

The refracted rays are constructed from the roots of the dispersion equation in $\mathscr{D}_2$ on Ω and proceed into $\mathscr{D}_2$ with increasing values of t. They satisfy the inequality

$$\mathbf{N} \cdot \dot{\mathbf{X}} \leqslant 0 \quad \text{on } \Omega. \tag{83}$$

From (81) we obtain the equivalent statement

$$[\mathbf{r}, Ar] \leqslant 0 \quad \text{on } \Omega. \tag{84}$$

† Distinct roots h must have distinct solutions α because of the constant multiplicity property given in §2·1.

‡ In our construction of the solution of an initial value problem for $\mathbf{u}$, the initial conditions have already been satisfied at $t = 0$. If any reflected or refracted rays were to leave the interface with *decreasing* t they would intersect the initial plane and then $\mathbf{u}$ would no longer satisfy the initial conditions of the problem.

4·4. *Initial values for the transport equations at the interface*

Under certain restrictions we can uniquely determine the initial values on Ω for the transport equations associated with the reflected and refracted rays when the incident wave is known. In this section we present these restrictions and describe the procedure for finding the necessary initial values.

Let us recall that for every distinct root $\omega = h(t, \mathbf{X}, \mathbf{k})$ of the dispersion relation in $\mathscr{D}_1$ there are q linearly independent null vectors $\mathbf{r}_j(t, \mathbf{X}, \mathbf{K})$ of the dispersion matrix. Now given a root $\omega = h$, each distinct α which satisfies (79) gives rise to a null space consisting of the vectors $\mathbf{r}_j(t, \mathbf{X}, \alpha\mathbf{N}+\mathbf{K}_0)$ $(j=1, \ldots, q)$. Suppose we take all these null vectors for all roots and all distinct α's in $\mathscr{D}_1$ and label them $\mathbf{R}^1_j$ $(j=1, \ldots, \mu_1)$. The preceding comments are equally valid in region $\mathscr{D}_2$. Therefore in a similar manner we obtain a set of μ_2 null vectors $\mathbf{R}^2_j$.

We assume that no incident ray, reflected ray, or refracted ray is tangent to the interface Ω. By (81) this property may be expressed algebraically by

$$[\mathbf{R}^i_j, A\mathbf{R}^i_j] \neq 0 \quad (i=1,2, j=1, \ldots, \mu_i). \tag{85}$$

The second assumption we make is that

$$\mu_1 = \mu_2 = \rho \tag{86}$$

where ρ is the rank of A.

THEOREM 4·1. At each point $\mathbf{X} \in \Omega$ where (85) and (86) are satisfied, the set of vectors $\mathbf{R}^i_j$ is linearly independent for each i.

This theorem is proved in appendix II.

We suppose that the incident wave originates in $\mathscr{D}_1$. Then, as indicated in §4·3, the reflected wave is also in $\mathscr{D}_1$ and the refracted wave is in $\mathscr{D}_2$. At the interface Ω we let $\mathbf{u}_1$ represent the sum of the incident and reflected waves

$$\mathbf{u}_1 \sim \sum_{\sigma_1} e^{i\lambda s_0} \sum_{j=0}^{\infty} (i\lambda)^{-j} \mathbf{z}^1_{j\sigma_1}. \tag{87}$$

Now we let $\mathbf{u}_2$ represent the sum of refracted waves at Ω,

$$\mathbf{u}_2 \sim \sum_{\sigma_2} e^{i\lambda s_0} \sum_{j=0}^{\infty} (i\lambda)^{-j} \mathbf{z}^2_{j\sigma_2}. \tag{88}$$

Then the jump $[\mathbf{u}]$ at Ω is given by

$$[\mathbf{u}] = \mathbf{u}_1 - \mathbf{u}_2 \sim e^{i\lambda s_0} \sum_{j=0}^{\infty} (i\lambda)^{-j} \Big(\sum_{\sigma_1} \mathbf{z}^1_{j\sigma_1} - \sum_{\sigma_2} \mathbf{z}^2_{j\sigma_2}\Big). \tag{89}$$

By imposing the interface condition (72) we obtain

$$\Big[\mathbf{Q}_\nu, \sum_{\sigma_1} \mathbf{z}^1_{j\sigma_1} - \sum_{\sigma_2} \mathbf{z}^2_{j\sigma_2}\Big] = 0 \quad (\nu=1, \ldots, \rho; j=0, 1, \ldots). \tag{90}$$

In order to find the initial values for the zero order transport equations on Ω we set $j = 0$ in (90). Since $\sum_{\sigma_1} \mathbf{z}^1_{0\sigma_1}$ can be expressed as a linear combination of the null vectors $\mathbf{R}^1_1, \ldots, \mathbf{R}^1_{\mu_1}$, we obtain

$$\sum_{\sigma_1} \mathbf{z}^1_{0\sigma_1} = \sum_{j=1}^{\mu_1} \sigma^1_j \mathbf{R}^1_j. \tag{91}$$

Similarly,
$$\sum_{\sigma_2} \mathbf{z}^2_{0\sigma_2} = \sum_{j=1}^{\mu_2} \sigma_j^2 \mathbf{R}_j^2. \tag{92}$$

Substitution of (91) and (92) into (90) results in

$$\left[\mathbf{Q}_\nu, \sum_{j=1}^{\mu_1} \sigma_j^1 \mathbf{R}_j^1 - \sum_{j=1}^{\mu_2} \sigma_j^2 \mathbf{R}_j^2\right] = 0 \quad (\nu = 1, \ldots, \rho). \tag{93}$$

Those coefficients σ_j^1 of the vectors $\mathbf{R}_j^1$ which satisfy $[\mathbf{R}_j^1, A\mathbf{R}_j^1] < 0$ are associated with the incident wave. Therefore they are known quantities. The coefficients σ_j^2 of the vectors $\mathbf{R}_j^2$ which satisfy $[\mathbf{R}_j^2, A\mathbf{R}_j^2] > 0$ are identically zero because we assume that the only waves in region $\mathscr{D}_2$ are refracted waves. The remaining coefficients in $\mathscr{D}_1$ and $\mathscr{D}_2$ are those corresponding to the reflected and refracted waves respectively. Let us suppose $[\mathbf{R}_j^1, A\mathbf{R}_j^1] > 0$ $(j = 1, \ldots, n_1)$ and $[\mathbf{R}_j^2, A\mathbf{R}_j^2] < 0$ $(j = 1, \ldots, n_2)$. Then (93) may be expressed as

$$\left[\mathbf{Q}_\nu, \sum_{j=1}^{n_1} \sigma_j^1 \mathbf{R}_j^1 - \sum_{j=1}^{n_2} \sigma_j^2 \mathbf{R}_j^2\right] = \mathscr{F}_\nu \quad (\nu = 1, \ldots, \rho), \tag{94}$$

where the quantities $\mathscr{F}_\nu$ are known from the incident wave. We observe that (94) is a linear algebraic system of ρ equations for $(n_1 + n_2)$ unknowns. It can be solved uniquely if $n_1 + n_2 = \rho$ and if the coefficient matrix in (94) is non-singular.

THEOREM 4·2. At each point $\mathbf{X} \in \Omega$ where (85) and (86) are satisfied, $n_1 + n_2 = \rho$ and the coefficient matrix in (94) is non-singular.

The proof of this theorem is found in appendix II. The existence and uniqueness of the quantities σ_j^1, σ_j^2 appearing in (94) is established by theorem 4·2. Thus we have derived the initial values for the zero order transport equations on the interface Ω under the various restrictions imposed above.

In principle, the initial values for the higher order transport equations can be found in a similar manner. We shall not attempt to do this here.

If (70) is isotropic, i.e. if the roots $\omega = h$ of the dispersion relation (10) depend on t, $\mathbf{X}$, and only the magnitude $k = \surd(\mathbf{K} \cdot \mathbf{K})$ of the vector $\mathbf{K}$, then properties (85) and (86) follow from conditions more simply understood in a physical sense. We first introduce an angle of incidence θ_0 by the equation

$$\cos\theta_0 = -(\mathbf{N} \cdot \mathbf{X}_0)/\surd(\mathbf{X}_0 \cdot \mathbf{X}_0), \tag{95}$$

where $\mathbf{X}_0$ is an incident ray.

THEOREM 4·3. Let (70) be isotropic. Then, given a point $\mathbf{X} \in \Omega$, if $|\omega_0|$ is sufficiently large where ω_0 is given by (74) and $|\theta_0|$ is sufficiently small, conditions (85) and (86) are satisfied.

This theorem is proved in appendix II.

5. Reflexion at a boundary

5·1. *Introduction, formulation of boundary condition*

Here we shall consider the initial boundary-value problem for the asymptotically conservative symmetric hyperbolic equation

$$A^0 \frac{\partial \mathbf{u}}{\partial t} + \Sigma A^\nu \frac{\partial \mathbf{u}}{\partial x_\nu} + \lambda B\mathbf{u} + C\mathbf{u} = \mathbf{f}(t, \mathbf{X}; \lambda). \tag{96}$$

Here
$$\mathbf{u}(0, \mathbf{X}; \lambda) = \mathbf{v}(\mathbf{X}; \lambda). \tag{97}$$

Let us suppose there is a smooth surface Ω in $\mathbf{X}$-space which is given parametrically by the equations
$$x_\nu = x_\nu(\xi_1, \ldots, \xi_{n-1}) \quad (\nu = 1, \ldots, n). \tag{98}$$

For simplicity we stipulate that the initial value $\mathbf{v}(\mathbf{X}, \lambda)$ and source function $\mathbf{f}(t, \mathbf{X}, \lambda)$ vanish in a neighbourhood of Ω. These assumptions eliminate the need for certain compatibility conditions.

The problem we wish to treat is similar to the interface problem considered in § 4. Our object is to find a formal asymptotic solution of (96), (97) which satisfies given conditions on the boundary Ω. On Ω we impose on the solution $\mathbf{u}$ a certain type of homogeneous boundary condition. The solution $\mathbf{u}$ is a vector in an m-dimensional vector space E^m. We introduce a subspace T of E^m with the following properties:

(i) the matrix A, given by (71), is non-positive over T, i.e.
$$[\mathbf{u}, A\mathbf{u}] \leqslant 0 \text{ for all } \mathbf{u} \in T,$$

(ii) T is maximal, i.e. the dimension of T is as large as the dimension of any subspace having property (i).

The boundary condition is then expressed as
$$\mathbf{u}(t, \mathbf{x}, \lambda) \in T \quad \text{for} \quad \mathbf{X} \in \Omega. \tag{99}$$

If $T^\perp$ is the orthogonal complement of T and $T^\perp$ is spanned by the vectors $\mathbf{Q}_1, \ldots, \mathbf{Q}_\mu$ then (99) is equivalent to
$$[\mathbf{Q}_j, \mathbf{u}] = 0, \quad j = 1 \ldots, \mu \quad \text{for} \quad \mathbf{X} \in \Omega. \tag{100}$$

This particular formulation of the boundary condition is sufficient to insure uniqueness of the (exact) solution $\mathbf{u}$ of (96), (99), (Courant & Hilbert 1962). The boundary condition for the field vectors in electromagnetic wave propagation at a perfect conductor is of the type given above (Lewis 1958).

Our procedure for this problem is similar to the procedure we followed in the interface problem. We suppose that a family of rays intersects the boundary Ω with increasing t. As before, these rays are called the incident rays and the wave associated with it, the incident wave. We construct additional waves, called reflected waves, emanating from Ω in such a way that the sum of the incident wave and reflected waves satisfy the given boundary conditions.

The construction of the reflected rays is given in detail in § 4·2. Here we have only one domain under consideration. The discussion of reflexion given in § 4·3 is also valid under the present circumstances. The only real difference between the two problems occurs in the determination of the initial values for the transport equations. That problem is presented in the next section.

5·2. *Initial values for the transport equations at the boundary*

The discussion in § 4·4 up to and including theorem 4·1 is valid for the present problem.

At the surface Ω the solution u of (96), (97) is the sum of the incident wave and reflected waves,
$$\mathbf{u} \sim \sum_\beta \mathrm{e}^{\mathrm{i}\lambda s_0} \sum_{j=0}^{\infty} (\mathrm{i}\lambda)^{-j} \mathbf{z}_{j\beta} \quad (\mathbf{X} \in \Omega). \tag{101}$$

We impose the boundary condition (100) on (101) to obtain

$$[\mathbf{Q}_i, \sum_\beta \mathbf{z}_{j\beta}] = 0 \quad (j=0,1,\ldots; i=1,\ldots,\mu). \tag{102}$$

To find the initial values for the zero order transport equation we set $j=0$ in (102). We recall that $\sum_\beta \mathbf{z}_{\rho\beta}$ can be expressed as a linear combination of the null vectors $\mathbf{R}_1, \ldots, \mathbf{R}_\rho$, i.e.

$$\sum_\beta \mathbf{z}_{\rho\beta} = \sum_{j=1}^{\rho} \sigma_i \mathbf{R}_i. \tag{103}$$

Inserting (103) into (102) we obtain

$$\sum_{j=1}^{\rho} \sigma_i [\mathbf{Q}_i, \mathbf{R}_j] = 0 \quad (i=1,\ldots,\mu, \mathbf{X} \in \Omega). \tag{104}$$

The coefficients σ_i associated with the incident wave are known at Ω. Therefore only the coefficients σ_i associated with reflected waves are to be determined. These are the coefficients for which $[\mathbf{R}_j, A\mathbf{R}_j] < 0$. Suppose the first d vectors $\mathbf{R}_1, \ldots, \mathbf{R}_j$ satisfy this inequality. Then we consider

$$\sum_{j=1}^{d} \sigma_i [\mathbf{Q}_i, \mathbf{R}_j] = F \quad (i=1,\ldots,\mu), \tag{105}$$

where F is a known quantity. This algebraic linear system has a unique solution if $d=\mu$ and the coefficient matrix is non-singular. In a manner similar to that of the interface problem, it can be shown that this is indeed the case. Therefore (105) can be solved uniquely and its solution gives the necessary inited data for the zero order transport equation.

Appendix I

Proof of theorem 2·1. Let $\mathbf{v}_\mu$ $(\mu=1,\ldots,q)$ be any basis of the null space of G^*. Form the $q \times q$ matrix $A = \{[\mathbf{v}_\mu, \mathbf{r}_j]\}$. If the matrix A is non-singular, $A^{-1} = \{a_{\mu j}\}$ exists and

$$A^{-1}A = \left\{\sum_{i=1}^{q} a_{\mu i}[\mathbf{v}_i, \mathbf{r}_j]\right\} = \left\{\left[\sum_{i=1}^{q} \bar{a}_{\mu i}\mathbf{v}_i, \mathbf{r}_j\right]\right\} = \{\delta_{uj}\} = I.$$

Set $\mathbf{p}_\mu = \sum_{i=1}^{q} \bar{a}_{\mu i}\mathbf{v}_i$. It is easily seen that the vectors $\mathbf{p}_\mu, \mu = 1, \ldots, q$, span the null space G^* and satisfy (22). We now demonstrate that the matrix A is nonsingular. From properties (i) and (ii), it follows that there exist m linearly independent vectors $\mathbf{r}_1, \ldots, \mathbf{r}_m$ such that

$$G\mathbf{r}_j = \begin{cases} 0 & (j=1,\ldots,q), \\ (h_j - h)\,\mathbf{r}_j & (j=q+1,\ldots,m), \end{cases} \tag{I 1}$$

where $h_j - h \neq 0$ $(j=q+1,\ldots,m)$. Hence $\mathbf{r}_j, j=1,\ldots,q$ span the null space $N(G)$ of G and $\mathbf{r}_j, j=q+1,\ldots,m$ span the range $R(G)$ of G. Thus the m-dimensional vector space E^m can be written as the direct sum

$$E^m = N(G) \oplus R(G). \tag{I 2}$$

If $\mathbf{v}$ belongs to $N(G^*)$, then $[\mathbf{v}, G\mathbf{r}] = [G^*\mathbf{v}, \mathbf{r}] = 0$. Hence $R(G) \perp N(G^*)$. Now suppose A is singular. This implies that there exist α_μ, not all zero, such that

$$\left[\sum_{\mu=1}^{q} \alpha_\mu \mathbf{v}_\mu, \mathbf{r}_j\right] = 0 \quad (j=1,\ldots,q). \tag{I 3}$$

Hence $\sum_{\mu=1}^{q} \alpha_\mu \mathbf{v}_\mu$ is orthogonal to $N(G)$. But $\sum_{\mu=1}^{q} \alpha_\mu \mathbf{v}_\mu$ is also orthogonal to $R(G)$. Thus it is orthogonal to every vector in E^m and must be the zero vector, i.e. $\alpha_\mu = 0$ ($\mu = 1, \ldots, g$). Since this is a contradiction, A must be non-singular.

Appendix II

1. *Derivation of interface condition*

We denote by region $\mathscr{D}_1$ those values of $\mathbf{X}$ for which $\psi(\mathbf{X}) > 0$ and by region $\mathscr{D}_2$ those values of $\mathbf{X}$ for which $\psi(\mathbf{X}) < 0$. Let us suppose that $\mathbf{u}_1$ is the solution of (70) in $\mathscr{D}_1$ and $\mathbf{u}_2$ is the solution in $\mathscr{D}_2$. Then the solution $\mathbf{u}$ of (70) may be expressed as

$$\mathbf{u}(t, \mathbf{X}) = \mathbf{u}_2(t, \mathbf{X}) + \eta[\psi(\mathbf{X})]\,[\mathbf{u}_1(t, \mathbf{X}) - \mathbf{u}_2(t, \mathbf{X})], \tag{II 1}$$

where $\eta[\psi]$ is the heaviside function. Furthermore, we may write

$$\left.\begin{aligned} A^0 &= A_2^0 + \eta[\psi(\mathbf{X})]\,(A_1^0 - A_2^0), \\ B &= B_2 + \eta[\psi(\mathbf{X})]\,(B_1 - B_2), \\ C &= C_2 + \eta[\psi(\mathbf{X})]\,(C_1 - C_2). \end{aligned}\right\} \tag{II 2}$$

Here A_2^0 is the matrix A^0 in region $\mathscr{D}_2$, etc. Substituting (II 2) into (70), we obtain

$$\begin{aligned} A_2^0 \frac{\partial \mathbf{u}_2}{\partial t} + \sum_{\nu=1}^{n} A^\nu \frac{\partial \mathbf{u}_2}{\partial x_\nu} + \lambda B_2 \mathbf{u}_2 + C_2 \mathbf{u}_2 + \eta(\psi) \Big[A_1^0 \frac{\partial \mathbf{u}_1}{\partial t} - A_2^0 \frac{\partial \mathbf{u}_2}{\partial t} \\ + \sum_{\nu=1}^{n} A^\nu \left(\frac{\partial \mathbf{u}_1}{\partial x_\nu} - \frac{\partial \mathbf{u}_2}{\partial x_\nu} \right) + \lambda(B_1 \mathbf{u}_1 - B_2 \mathbf{u}_2) + C_1 \mathbf{u}_1 - C_2 \mathbf{u}_2 \Big] \\ + \delta(\psi)\,|\mathrm{grad}\,\psi|\,A[\mathbf{u}_1 - \mathbf{u}_2] = \mathbf{f}(t, \mathbf{X}, \lambda), \end{aligned} \tag{II 3}$$

where $\delta(\psi) = \eta'(\psi)$ is the delta function and A is given by (71). Since $\mathbf{f}(t, \mathbf{X}; \lambda)$ vanishes in a neighbourhood of Ω, (II 3) implies that

$$A[\mathbf{u}_1 - \mathbf{u}_2] = 0 \quad \text{at } \Omega. \tag{II 4}$$

2. *Proofs of theorems* 4·1 *and* 4·2

We list the following definitions for the sake of convenience.

E^m: a vector space of dimension m. (II 5)

$N = (\eta_\nu)$: the unit normal to the interface Ω. (II 6)

A: matrix defined by $A = \sum_{\nu=1}^{n} \eta_\nu A^\nu$. (II 7)

$\mathbf{Q}_1, \ldots, \mathbf{Q}_\rho$: a basis for the orthogonal complement of the null space of A, ρ is the rank of A. (II 8)

$$M = \begin{pmatrix} [\mathbf{Q}_1, \mathbf{R}_1^1] \ldots [\mathbf{Q}_1, \mathbf{R}_{n_1}^1]\; [\mathbf{Q}_1, \mathbf{R}_1^2] \ldots [\mathbf{Q}_1, \mathbf{R}_{n_2}^2] \\ \cdot \quad \ldots \quad \cdot \qquad \cdot \quad \ldots \quad \cdot \\ [\mathbf{Q}_\rho, \mathbf{R}_1^1] \ldots [\mathbf{Q}_\rho, \mathbf{R}_{n_1}^1]\; [\mathbf{Q}_\rho, \mathbf{R}_1^2] \ldots [\mathbf{Q}_\rho, \mathbf{R}_{n_2}^2] \end{pmatrix}$$

: the matrix coefficient of the linear algebraic system (94). (II 9)

Proof of theorem 4·1.

The proof of this theorem requires the following lemma.

Lemma 4·1. $[\mathbf{R}_\nu^i, A\mathbf{R}_j^i] = 0$ on Ω if $\nu \neq j$ ($i = 1, 2$). (II 10)

Proof. We may delete the superscript i because the proof is independent of the regions $\mathscr{D}_1$ and $\mathscr{D}_2$. From the definition of the dispersion matrix G and (78), it is an easy calculation to show that

$$(\alpha_j A + F)\, \mathbf{R}_j = 0 \tag{II 11}$$

and

$$(\alpha_\nu A + F)\, \mathbf{R}_\nu = 0, \tag{II 12}$$

where $F = \sum_{\mu=1}^{n} k_{\nu 0} A^\nu - \mathrm{i}B - \omega_0 A^0$. Let us first suppose that $\mathbf{R}_j$ and $\mathbf{R}_\nu$ are associated with distinct roots of the dispersion relation. Then α_j and α_ν are not equal because of the property of constant multiplicity of roots. From (II 11) and (II 12) we obtain the equation

$$0 = [\mathbf{R}_\nu, (\alpha_j A + F)\, \mathbf{R}_j] = (\alpha_\nu - \alpha_j)\, [\mathbf{R}_\nu, A\mathbf{R}_j]. \tag{II 13}$$

Since $\alpha_\nu - \alpha_j \neq 0$ we must conclude that

$$[\mathbf{R}_\nu, A\mathbf{R}_j] = 0 \quad (\nu \neq j) \tag{II 14}$$

if $\mathbf{R}_j$ and $\mathbf{R}_\nu$ are associated with distinct roots. If $\mathbf{R}_j$ and $\mathbf{R}_\nu$ are associated with the same root but α_ν and α_j are distinct then (II 14) still holds. If α_ν and α_j are equal but $\mathbf{R}_j$ and $\mathbf{R}_\nu$ are two linearly independent vectors in the null space of a root, then (II 14) is valid by (23) because

$$[\mathbf{R}_\nu, A\mathbf{R}_j] = \sum_{\mu=1}^{n} \eta_\nu [\mathbf{R}_\nu, A^\mu \mathbf{R}_j] = 0 \quad (\nu \neq j). \tag{II 15}$$

Now we are ready to complete the proof of theorem 4·1. Let us assume that $\sum_{j=1}^{\mu_i} \beta_j \mathbf{R}_j^i = 0$. Then, by lemma 4·1,

$$\sum_{j=1}^{\mu_i} \beta_j [\mathbf{R}_\nu^i, A\mathbf{R}_j^i] = \beta_\nu [\mathbf{R}_\nu^i, A\mathbf{R}_\nu^i] = 0 \quad (\nu = 1, \ldots, \mu_i). \tag{II 16}$$

Since $[\mathbf{R}_\nu^i, A\mathbf{R}_\nu^i] \neq 0$ on Ω by (85), (II 16) implies $\beta_\nu = 0$ $(\nu = 1, \ldots, \mu_i)$. Hence the vectors $\mathbf{R}_j^i$ are linearly independent for $j = 1$ or $j = 2$.

Proof of theorem 4·2

LEMMA 4·2. Let V_i be the subspace spanned by $\mathbf{R}_j^i$ where $[\mathbf{R}_j^i, A\mathbf{R}_j^i] > 0$, W_i be the subspace spanned by $\mathbf{R}_j^i$ where $[\mathbf{R}_j^i, A\mathbf{R}_j^i] < 0$, and N be the null space of the matrix A. Then

(i) $E^m = V_i \oplus W_k \oplus N \quad (i, k = 1, 2).$ (II 17)

(ii) $\dim V_1 = \dim V_2$, $\dim W_1 = \dim W_2$. (II 18)

Proof. Since the vectors $\mathbf{R}_j^i$ $(j = 1, \ldots, \rho)$ are linearly independent, it follows immediately that the subspaces V_i and W_i are disjoint for $i = 1$ and $i = 2$. If a vector $\mathbf{v} \in V_i$ then it can be shown, by lemma 4·1, that $[\mathbf{v}, A\mathbf{v}] \geq 0$. The equality holds only if $\mathbf{v} = 0$. Similarly, if $\mathbf{v} \in W_i$, $[\mathbf{v}, A\mathbf{v}] \leq 0$ where the equality holds only if $\mathbf{v} = 0$. From these properties of V_i and W_i it follows that the null space N of A, V_i, and W_i are disjoint. Since $\dim (V_i \oplus W_i) = \rho$ and $\dim N = m - \rho$, we obtain the result

$$E^m = V_i \oplus W_i \oplus N \quad (i = 1, 2). \tag{II 19}$$

Now $\dim V_i \leq p$ where p is the number of positive eigenvalues of A and $\dim (W_i \oplus N) \leq m - p$. But $\dim V_i + \dim (W_i \oplus N) = m$. Hence $\dim V_1 = \dim V_2 = p$ and $\dim W_1 = \dim W_2 = m - p$.

It is easily seen that the subspaces V_1 and W_2 are disjoint since A is positive definite on V_1 and negative definite on W_2. Since $\dim V_1 + \dim W_2 + \dim N = m$, we obtain

$$E^m = V_1 \oplus W_2 \oplus N. \tag{II 20}$$

Similarly

$$E^m = V_2 \oplus W_1 \oplus N. \tag{II 21}$$

From lemma 4·2 we find that (93) is an algebraic system of ρ equations for ρ unknowns since $n_1 = \dim V_1$ and $n_2 = \dim W_2$ where n_1 and n_2 appear in (94) and $\dim V_1 + \dim W_2 = \rho$.

Lemma 4·3. The matrix M, given by (II 9), is non-singular.

Proof. Assume the coefficient matrix is singular. This implies that there exist numbers $\beta_1, \ldots, \beta_\rho$, not all zero, such that

$$\begin{aligned} [\beta_1 \mathbf{Q}_1 + \ldots + \beta_\rho \mathbf{Q}_\rho, \mathbf{R}_j^1] &= 0 \quad (j = 1, \ldots, n_1 = \dim V_1), \\ [\beta_1 \mathbf{Q}_1 + \ldots + \beta_\rho \mathbf{Q}_\rho, \mathbf{R}_j^2] &= 0 \quad (j = 1, \ldots, n_2 = \dim W_2). \end{aligned} \tag{II 22}$$

There exists a non-zero vector $\mathbf{v} = \beta_1 \mathbf{Q}_1 + \ldots + \beta_\rho \mathbf{Q}_\rho$ which belongs to the orthogonal complement of the null space of A by (II 8) and which is orthogonal to every vector in the subspace $V_1 \oplus W_2$. Since $\mathbf{v}$ is orthogonal to N and $E^m = V_1 \oplus W_2 \oplus N$, $\mathbf{v}$ is orthogonal to every vector in E^m. Hence it must be the zero vector. This is a contradiction and thus the lemma follows.

3. *Proof of theorem* 4·3

This proof involves some properties of isotropic asymptotically conservative symmetric hyperbolic equations which we now present. Let us set $k_\nu = k\alpha_\nu$, $\sum_{\nu=1}^{n} \alpha_\nu^2 = 1$, and substitute into the dispersion matrix G given by (9). The result is

$$G = k\mathscr{A} - \mathrm{i}B - h(t, \mathbf{X}, k)\, A^0, \tag{II 23}$$

where $\mathscr{A} = \sum_{\nu=1}^{n} \alpha_\nu A^\nu$ and h is any root of the dispersion relation. We consider

$$\det G = \det [k\mathscr{A} - \mathrm{i}B - h(t, \mathbf{X}, k)\, A^0] = 0. \tag{II 24}$$

Dividing (II 24) by k^m we obtain

$$\det \left[\mathscr{A} - \frac{\mathrm{i}B}{k} - \frac{h(t, \mathbf{X}, k)}{k} A^0 \right] = 0. \tag{II 25}$$

Letting k tend to infinity in (II 25) yields

$$\det [\mathscr{A} - \hat{h}(t, \mathbf{X})\, A^0] = 0, \tag{II 26}$$

where $\hat{h}(t, \mathbf{X}) = \lim_{k \to \infty} [h(t, \mathbf{X}, k)/k]$. From (II 26) it is apparent that $\hat{h}(t, \mathbf{X})$ is finite.

Lemma 5·4. Every root $\omega = h(t, \mathbf{X}, k)$ of the dispersion relation (10) can be expressed as

$$h(t, \mathbf{X}, k) = k \left\{ \hat{h}(t, \mathbf{X}) - \int_k^\infty \frac{[\mathbf{r}(t, \mathbf{X}, \xi \mathbf{K}/k), \mathrm{i}B\mathbf{r}(t, \mathbf{X}, \xi \mathbf{K}/k)]}{\xi^2} \mathrm{d}\xi \right\}. \tag{II 27}$$

Here $\mathbf{r}$ is a null vector of $G(t, \mathbf{X}, h, \mathbf{K})$ and $[\mathbf{r}, A^0\mathbf{r}] = 1$.

Proof. We have

$$[k\mathscr{A} - \mathrm{i}B - h(t, \mathbf{X}, k)\, A^0]\, \mathbf{r}(t, \mathbf{X}, \mathbf{K}) = 0. \tag{II 28}$$

Taking the inner product of $\mathbf{r}(t, \mathbf{X}, \mathbf{K})$ and (II 28) we obtain

$$k[\mathbf{r}, \mathscr{A}\mathbf{r}] - [\mathbf{r}, \mathrm{i}B\mathbf{r}] - h[\mathbf{r}, A^0\mathbf{r}] = 0. \tag{II 29}$$

Now it follows from the basic identity (23) and the definition of $\mathscr{A}$ that

$$[\mathbf{r}, \mathscr{A}\mathbf{r}] = \sum_{\nu=1}^{n} \alpha_\nu g_\nu = \sum_{\nu=1}^{n} \alpha_\nu \frac{\partial h}{\partial k_\nu} = \frac{\mathrm{d}h}{\mathrm{d}k}. \tag{II 30}$$

Since $[r, A^0\mathbf{r}] = 1$, we obtain from (II 29) and (II 30)

$$\frac{\mathrm{d}h}{\mathrm{d}k} - \frac{h}{k} = \frac{[\mathbf{r}, \mathrm{i}B\mathbf{r}]}{k}. \tag{II 31}$$

A solution of this first order ordinary differential equation is

$$h(t, \mathbf{X}, \mathbf{k}) = k\left[\text{const.} + \int_1^k \frac{[\mathbf{r}(t, \mathbf{X}, \xi\mathbf{K}/k), \mathrm{i}B\mathbf{r}(t, \mathbf{X}, \xi\mathbf{K}/k)]}{\xi^2}\,\mathrm{d}\xi\right]. \tag{II 32}$$

The constant of integration is found as follows. Since

$$\lim_{k\to\infty} \frac{h}{k} = \text{const.} + \int_1^\infty \frac{[\mathbf{r}, \mathrm{i}B\mathbf{r}]}{\xi^2}\,\mathrm{d}\xi = \hat{h},$$

we obtain

$$\text{const.} = \hat{h} - \int_1^\infty \frac{[\mathbf{r}, \mathrm{i}B\mathbf{r}]}{\xi^2}\,\mathrm{d}\xi.$$

Insertion of this constant into (II 32) yields (II 27).

LEMMA 4·5. Let $b_{\text{min.}}$ and $b_{\text{max.}}$ be the smallest and greatest eigenvalue respectively of the hermitian matrix iB. Let $a^0_{\text{min.}}$ and $a^0_{\text{max.}}$ be the smallest and greatest eigenvalue respectively of the positive definite matrix A^0. Then

$$\frac{b_{\text{min.}}}{a^0_{\text{max.}}} \leqslant k\hat{h}(t, \mathbf{X}) - h(t, \mathbf{X}, k) \leqslant \frac{b_{\text{max.}}}{a^0_{\text{min.}}}. \tag{II 33}$$

Proof. From the extremal properties of eigenvalues of hermitian matrices we have the inequalities

$$b_{\text{min.}}[\mathbf{r}, \mathbf{r}] \leqslant [\mathbf{r}, \mathrm{i}B\mathbf{r}] \leqslant b_{\text{max.}}[\mathbf{r}, \mathbf{r}] \tag{II 34}$$

and

$$0 < a^0_{\text{min.}}[\mathbf{r}, \mathbf{r}] \leqslant [\mathbf{r}, A^0\mathbf{r}] = 1 \leqslant a^0_{\text{max.}}[\mathbf{r}, \mathbf{r}]. \tag{II 35}$$

From (II 34) and (II 35) it follows that

$$\frac{b_{\text{min.}}}{a^0_{\text{max.}}} \leqslant [\mathbf{r}, \mathrm{i}B\mathbf{r}] \leqslant \frac{b_{\text{max.}}}{a^0_{\text{min.}}}. \tag{II 36}$$

By (II 27) we obtain

$$k\left\{\hat{h} - \frac{b_{\text{max.}}}{a^0_{\text{min.}}}\int_k^\infty \xi^{-2}\,\mathrm{d}\xi\right\} \leqslant h \leqslant k\left\{\hat{h} - \frac{b_{\text{min.}}}{a^0_{\text{max.}}}\int_k^\infty \xi^{-2}\,\mathrm{d}\xi\right\}. \tag{II 37}$$

Evaluation of the integrals and some rearrangement yields II 33).

If the root $h(t, \mathbf{X}, k)$ has multiplicity q, then $\hat{h}(t, \mathbf{X})$ also has multiplicity q because, by property (iii) of § 2·1, q is independent of $\mathbf{K}$. Since the sum of all the multiplicities of the roots h of the dispersion relation is equal to m by property (ii) of § 2·1, it then follows that the sum of all the multiplicities of the limits $\hat{h}(\mathrm{t}, \mathbf{X})$ is also m. This immediately implies that every root of (II 26) is equal to $\lim_{k\to\infty} h(t, \mathbf{X}, k)/k$ for some $h(t, \mathbf{X}, k)$ of the dispersion relation.

Since all the limits $\hat{h}(t, \mathbf{X})$ are independent of the direction numbers $\alpha_1, \ldots, \alpha_n$, we may replace the matrix $\mathscr{A}$ by $-\mathscr{A}$ and still obtain the same set of roots of (II 26). Hence we conclude that for every root $\hat{h}(t, \mathbf{X})$, $-\hat{h}(t, \mathbf{X})$ is also a root. It is easily seen that the rank of $\mathscr{A}$ is equal to the total multiplicity of the non-zero roots $\hat{h}$ of (II 26). Therefore we find that the rank of $\mathscr{A}$ is independent of $\alpha_1, \ldots \alpha_n$ and is an even integer. For $\alpha_\nu = \eta_\nu$ where $\mathbf{N} = (\eta_\nu)$ is the unit normal to Ω, $\mathscr{A} = A$. Therefore rank $\mathscr{A}$ = rank $A = \rho$.

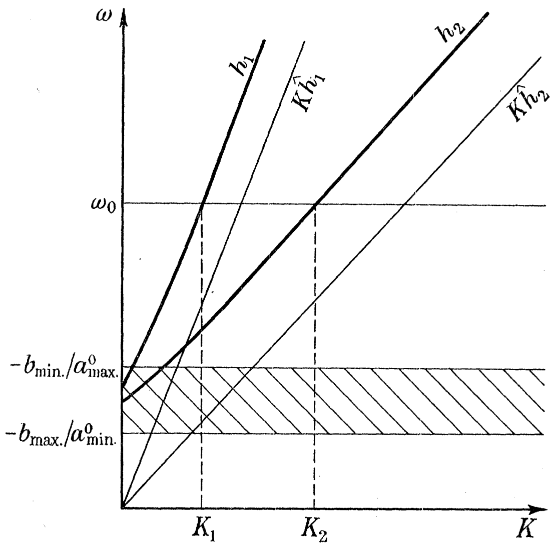

FIGURE 1

From (11) we observe that if $\hat{h} = 0$ then the root h remains in the domain

$$-b_{\max.}/a^0_{\min.} \leqslant h \leqslant -b_{\min.}/a^0_{\max.}$$

(see figure 1). If $\hat{h} > 0$ then the root h tends to infinity in the domain $\omega > -b_{\min.}/a^0_{\max.}$ and if $\hat{h} < 0$, h tends to minus infinity in the domain $\omega < -b_{\max.}/a^0_{\min.}$. In the following discussion we consider only those roots h which are unbounded.

Since the total multiplicity of all positive (negative) roots $\hat{h}$ of (II 26) must be $\frac{1}{2}\rho$ it follows that the total multiplicity of the roots h of the dispersion relation which tend to ∞ $(-\infty)$ is $\frac{1}{2}\rho$.

Let us first consider domain $\mathscr{D}_1$. We assume that ω_0, given by (74), satisfies

$$\omega_0 > -b_{1\,\min.}/a^0_{1\,\max.}. \tag{II 38}$$

Here $b_{1\,\min.}$ is the smallest eigenvalue of the matrix iB defined in $\mathscr{D}_1$ and $a^0_{1\,\max.}$ is the greatest eigenvalue of the matrix A^0 defined in $\mathscr{D}_1$. We now show that if $|\theta_0|$, where θ_0 is given by (95), is sufficiently small then, for each distinct unbounded root h, which tends to $+\infty$, of the dispersion relation in $\mathscr{D}_1$, there are exactly two solutions α of (79).

From (II 31) it follows that $\mathrm{d}h/\mathrm{d}k > 0$ for $h > -b_{1\,\min.}/a^0_{1\,\max.}$. Therefore in the domain $\omega > -b_{1\,\min.}/a^0_{1\,\max.}$, every unbounded root is monotonically increasing to $+\infty$. Hence there exists one and only one real value k for each distinct root such that $h(t, \mathbf{X}, k) = \omega_0$ (see figure 1). For each such k we can find two solutions α in the following manner. We set

$$k^2 = \alpha^2 + 2\alpha \mathbf{N}\cdot\mathbf{K}_0 + k_0^2. \tag{II 39}$$

Solving this quadratic equation for α we obtain

$$\alpha = -(\mathbf{N}\cdot\mathbf{K}_0) \pm \surd\{(\mathbf{N}\cdot\mathbf{K}_0)^2 - (k_0^2 - k^2)\}. \tag{II 40}$$

In the isotropic case it follows from the ray equations (14) that

$$\dot{\mathbf{X}} = \left(\frac{1}{k}\frac{\partial h}{\partial k}\right)\mathbf{K} \tag{II 41}$$

for each root h of the dispersion relation. From (II 41) and (95) we obtain

$$(\mathbf{N}\cdot\mathbf{K}_0) = -\operatorname{sgn}(\partial h_0/\partial k)\,k_0\cos\theta_0. \tag{II 42}$$

Here h_0 is the root gerenting the incident ray $\mathbf{X}_0$. Substitution of (II 42) in (II 40) yields

$$\alpha = \operatorname{sgn}(\partial h_0/\partial k)\,k_0\cos\theta_0 \pm \surd\{k^2 - k_0^2\sin^2\theta_0\}. \tag{II 43}$$

If

$$\sin^2\theta_0 < k^2/k_0^2 \tag{II 44}$$

for each distinct increasing unbound root, (II 43) gives us two real solutions α for each such root.

Let us now consider domain $\mathscr{D}_2$ in which $b_{2\,\mathrm{min.}}$ and $a^0_{2\,\mathrm{max.}}$ are defined. We assume, in addition to (II 38), that

$$\omega_0 > -b_{2\,\mathrm{min.}}/a^0_{2\,\mathrm{max.}}. \tag{II 45}$$

Proceeding again in the manner outlined above we find that there are exactly two solutions α of (79) for each increasing unbounded root of the dispersion relation in $\mathscr{D}_2$. Identical results can be obtained for

$$\omega_0 < \min_{i=1,2}\,(-b_{i\,\mathrm{max.}}/a^0_{i\,\mathrm{min.}}). \tag{II 46}$$

and for

$$-b_{i\,\mathrm{min.}}/a^0_{i\,\mathrm{max.}} < \omega_0 < -b_{j\,\mathrm{max.}}/a^0_{j\,\mathrm{min.}} \quad (i \neq j,\ i,j = 1,2) \tag{II 47}$$

if the latter inequality is meaningful.

Now we have completed all the prerequisites in order to complete the proof of theorem 4·3. Let us first derive property (85). Let $\mathbf{R}$ be any null vector corresponding to an unbounded h. Then, by (II 41),

$$[\mathbf{R}, A\mathbf{R}] = \mathbf{N}\cdot\dot{\mathbf{X}} = \left(\frac{1}{k}\frac{\partial h}{\partial k}\right)\mathbf{N}\cdot\mathbf{K}. \tag{II 48}$$

By (78) and (II 43),

$$\mathbf{N}\cdot\mathbf{K} = \pm\surd(k^2 - k_0^2\sin^2\theta_0) \neq 0. \tag{II 49}$$

Because we consider the unbounded roots in the regions where they are monotonic, $\partial h/\partial k \neq 0$. Hence $[\mathbf{R}, A\mathbf{R}] \neq 0$ and property (85) is satisfied.

To derive property (86) we first note that μ_i $(i = 1, 2)$ is equal to the sum of the products of the number of solutions α of each distinct root and the multiplicity of each distinct root of the dispersion relation in $\mathscr{D}_i$. The number of solutions α of each distinct root having solutions is two and the sum of the multiplicities of the distinct roots having solutions is $\frac{1}{2}\rho$. Hence $\mu_1 = \mu_2 = \rho$.

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